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Classification of Quadratic Forms

SEMINAR: THE ARITHMETICS OF THE HYPERBOLIC PLANE

1. General Definitions and Results

Note 1.1. If not stated else, $Q(x, y)$ always denotes a quadratic form with integer coefficients.

Definition 1.2. Q is called

(i) *hyperbolic*, if $Q(x, y)$ takes on both positive and negative values, but not zero,

(ii) 0-*hyperbolic*, if $Q(x, y)$ takes on both positive and negative values, and zero,

(iii) *elliptic*, if $Q(x, y)$ takes on only positive values or only negative values,

(iv) *parabolic*, if $Q(x, y)$ takes on only non-negative values or only non-positive values, and takes on zero.

Definition 1.3. If $Q(x, y) = ax^2 + bxy + cy^2$, then

$$\Delta = \Delta(Q) = b^2 - 4ac$$

is called the *discriminant* of Q.

Theorem 1.4. *Q can be characterised via its discriminant:*

(i) If $\Delta > 0$ is no square, then Q is hyperbolic.

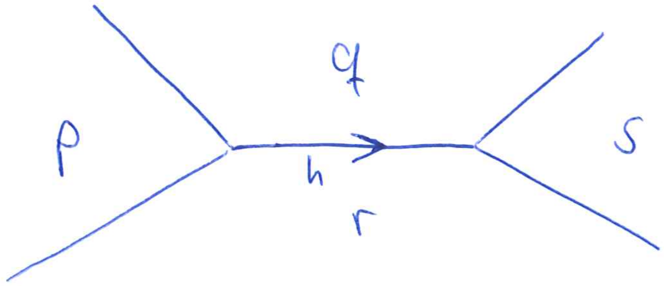
(ii) If $\Delta > 0$ is a square, then Q is 0-hyperbolic.

(iii) If $\Delta < 0$, then Q is elliptic.

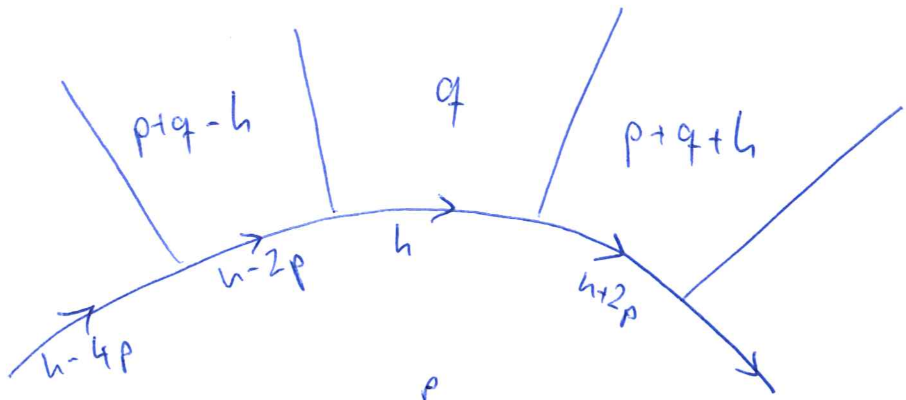
(iv) If $\Delta = 0$, then Q is parabolic.

Definition 1.5. Given an edge in the topograph of Q, we have an arithmetic progression $p, q+r, s$. By orientating the edge, we can label it by the common increment of the progression:

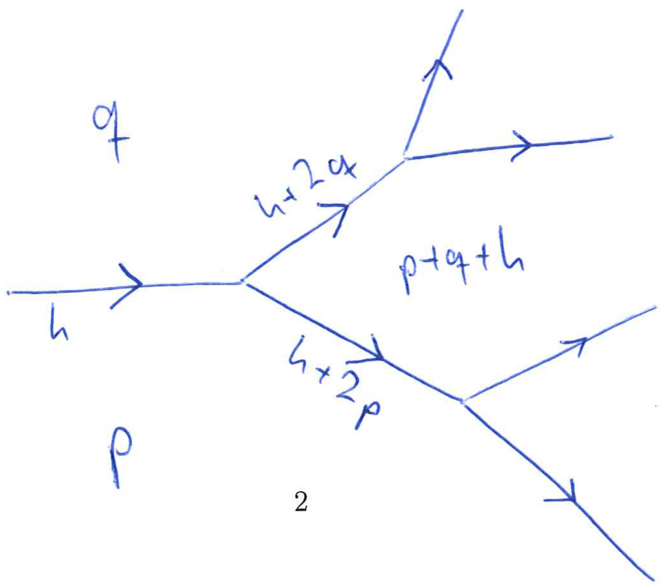
$$h = s - (q + r) = (q + r) - p$$



Lemma 1.6 (Second Arithmetic Progression Rule). *The values of the edges along the border of a region form an arithmetic progression.*



Lemma 1.7 (Monotocity Rule). *If $h, p, q > 0$, then the proceeding edges (oriented like in the picture) and the next adjacent region also have positive values. Inductively, all proceeding edges and regions "in this direction" have positive value. The analogous statement for negative values ist also true.*



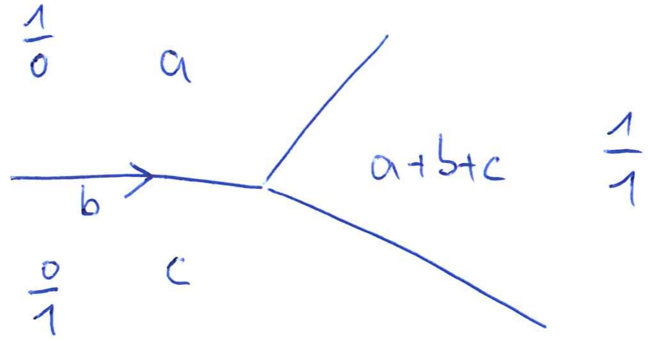
Lemma 1.8. *For any edge in the topograph of Q labeled h with adjacent regions labeled p, q, we have*

$$\Delta(Q) = h^2 - 4pq.$$

Proof. We prove this by induction. Suppose, Q is given by $Q(x, y) = ax^2 + bxy + cy^2$, then we have:

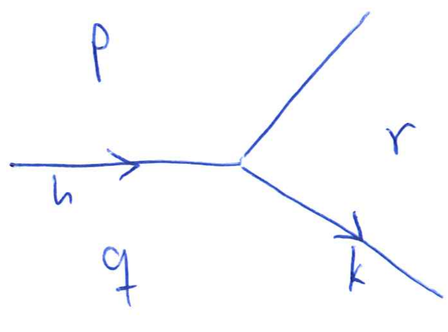
$$Q(1, 0) = a$$
$$Q(0, 1) = c$$
$$Q(1, 1) = a + b + c$$

From this we get, that the statement is true for the edge between the regions $\frac{1}{0}$ and $\frac{0}{1}$.



Now consider two joining edges, and suppose that $\Delta(Q) = h^2 - 4pq$. By the arithmetic progression rules we have

$$r = p + q + h$$
$$k = h + 2q.$$



Thus we get

$$k^2 - 4qr = (h + 2q)^2 - 4q(p + q + h) = h^2 - 4pq = \Delta(Q).$$

So, by induction over the edges of the (connected) topograph, we get that the statement is true for all edges. □

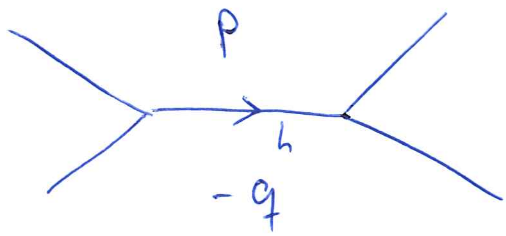
2. Hyperbolic Forms

Definition 2.1. An edge in the topograph of Q is called a *seperating edge*, if the two adjacent regions have opposite signs.

Theorem 2.2. *Let Q be hyperbolic. Then the seperating edges form a line infinite in both directions and the topograph is periodic along that line.*

Proof. First of all, since Q is hyperbolic, there exist regions of opposite signs. Moving along a path joining two such regions, one finds a seperating edge, since there are no regions with value zero.
Now, starting from at a fixed seperating edge, in each direction, exactly one of the proceeding edges is again a seperating edge. In this way, one gets an infinite line of seperating edges.

Moving off this seperating line, by monotocity the values of the regions and edges will be strictly positive on one side of the line and strictly negative on the other side. So there are no more seperating edges.
For periodicity, consider a seperating edge, $p, q > 0$.



Then we have $\Delta(Q) = h^2 + 4pq$, so

$$|h| < \sqrt{\Delta}$$
$$p, q \leq \Delta/4$$

So there are only finitely many possibilities for the triple (h, p, q) and it follows, that there exist two seperating edges having the same values and same adjacent region values. But the topograph is completely determined by the values h, p, q at a single edge, so this implies, that the topograph is periodic along the seperating line. □

Corollary 2.3. *If Q is hyperbolic and the equation $Q(x, y) = n$ has one integer solution, then it has infinitely many integer solutions.*

Proof. Let (x, y) be a pair of integers, such that $Q(x, y) = n$. If (x, y) is a primitive pair, then the corresponding region in the topograph of Q has label n. Then, by the periodicity of the topograph there are infinitely many primitive solutions to the equation. If (x, y) is not primitive, then $(x, y) = m(x', y')$ with (x', y') primitive and $Q(x', y') = n/m^2$. But this equation has infinitely many solutions by the above and the m-multiples of these solutions are solutions to the original equation. □

Lemma 2.4. *If Q is hyperbolic, then $\Delta(Q) > 0$.*

Proof. Take a seperating edge with value h and adjacent regions labeled p, q. Then $pq < 0$ and $h^2 \geq 0$, so $\Delta = h^2 - 4pq > 0$ □

Theorem 2.5. *Every "periodic pattern" appears as the seperator line of some hyperbolic form.*

Proof. Suppose the pattern is given as a line in the topograph. There is a linear fractional transformation, that translates the pattern by its symmetry, suppose this transformation is given by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integers a, b, c, d. Regarding this transformation as an isometry of the hyperbolic plane, it is hyperbolic itself, so it has two fixed points in $\mathbb{R} - \mathbb{Q}$. These are exactly the limits of the rationals along the pattern, when moving to the left, resp. moving to the right. For a fixed point z we have:

$$z = \frac{az + b}{cz + d}$$

$$cz^2 + (d - a)z - b = 0$$

Now, consider the quadratic form $Q(x, y) = cx^2 + (d - a)xy - by^2$. It is hyperbolic and has the desired seperator line. □

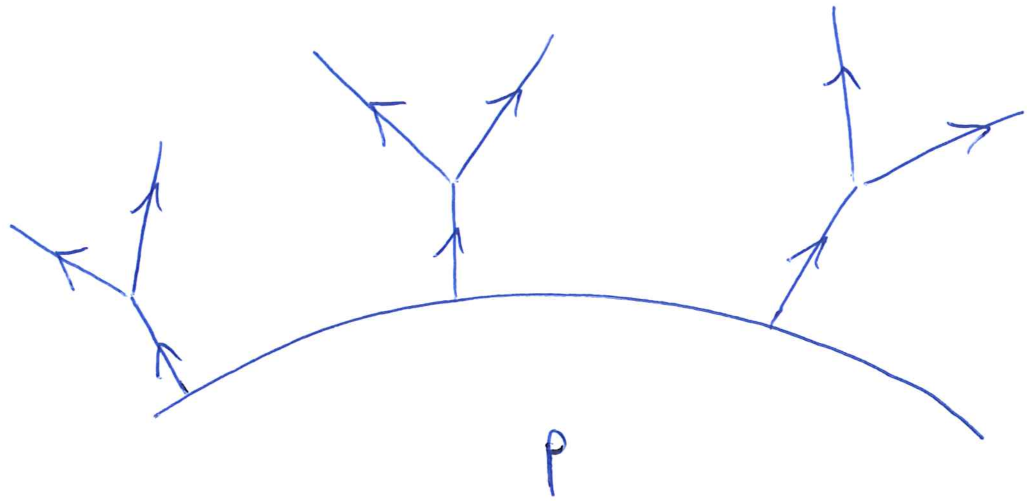
Note 2.6. There is a connection to continued fractions, since the fixed points can be expressed as periodic continued fractions by using the given pattern.

3. Elliptic Forms

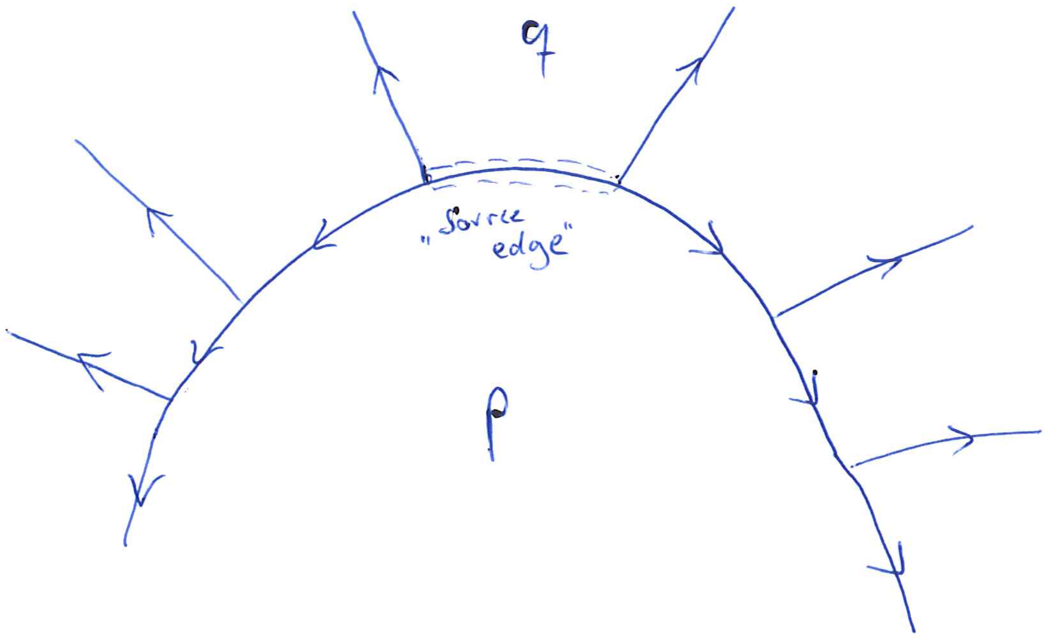
Note 3.1. We restrict ourselves to elliptic forms which take on only positive values, since the theory for elliptic curves with negative values is completely analogous.

Lemma 3.2. *If Q is elliptic, then $\Delta(Q) < 0$.*

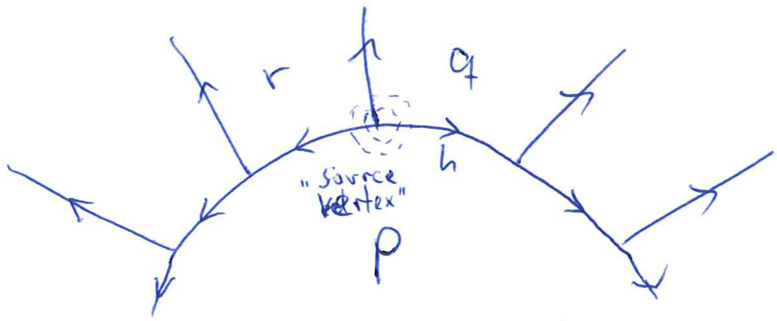
Proof. Choose orientations of the non-zero edges of the topograph, such that all of its values are positive. Now let p be the minimum value of Q and consider a region labeled p. All edges heading away from this region are also oriented away from this region by arithmetic progression. By monotocity, all edges farther away are oriented away from the region.



Now distinguish two possibilities:
(I) If some edge adjacent to the region has value 0, then by arithmetic progression, all other border edges of the region are oriented away from that edge. This unique edge with value 0 is called a *source edge*.



(II) If there is no border edge with value 0, then by arithmetic progression, there is one border vertex, where the orientation of the border-edges change. This unique vertex is called a *source vertex.*



In the first case, we consider the source edge and get $\Delta = 0^2 - 4pq < 0$, since $p, q > 0$. In the second case consider an edge at the source vertex. Then we have

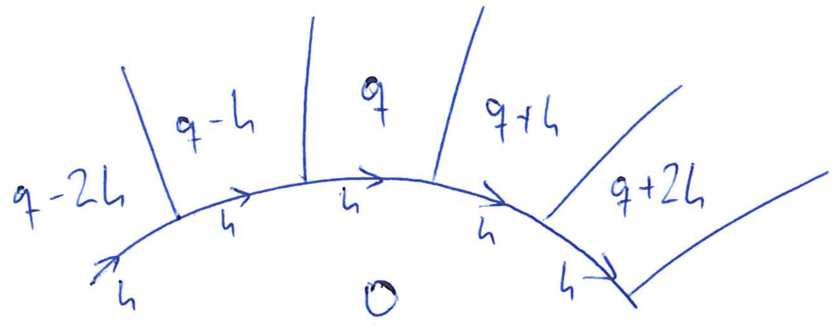
$$\Delta = h^2 - 4pq = (p + q - r)^2 - 4pq = p(p - q - r) + q(q - p - r) + r(r - p - q).$$

But since all of the edges are oriented away from the source vertex, $p < q + r$ and the same is true for permutations of p, q, r. So $\Delta < 0$ in this case. $\square$

4. Parabolic and 0-Hyperbolic Forms

Lemma 4.1. *If Q is parabolic, then $\Delta(Q) = 0$. If Q is 0-hyperbolic, then $\Delta(Q)$ is a positive square.*

Proof. If Q is either parabolic or 0-hyperbolic, then there exists a region in the topograph of Q with label 0. Then the values of the bordering edges of this region are all the same (if oriented in the same way) by arithmetic progression. Let h be this value. Then we have $\Delta = h^2$, which is a square.



If $h \neq 0$, then by arithmetic progression, the adjacent regions to the 0-region take on both signs, so Q is 0-hyperbolic.
Conversely, if $h = 0$, then all the adjacent regions to the 0-region take on the same value, which is either positive, negative or zero. But then by monotocity, the regions further away have the same sign. So Q is parabolic in this case. $\square$

Lemma 4.2. *If $\Delta(Q)$ is a square, then Q is either* 0*-hyperbolic or parabolic.*

Proof. Let Q be given by $Q(x, y) = ax^2 + bxy + cy^2$.
If $a = 0$, then $Q(1, 0) = 0$, so we are done. Now suppose $a \neq 0$. Then the Equation $ax^2 + bx + c = 0$ has roots $x = (-b \pm \sqrt{b^2 - 4ac})/2a$. But $b^2 - 4ac$ is a square, so the quadratic equation has a rational solution $x = p/q$ where p, q are coprime integers. This already implies, that $ap^2 + bpq + cq^2 = 0$, so $Q(p, q) = 0$. □

This completes the proof of the Theorem above about the characterisation of quadratic forms.

References

[1] A. Hatcher, *Topology of Numbers.*